Invertible sheaves (Har II 6)

Recall: An invertible sheaf on a scheme X is a locally free sheaf of rank one.

If $\mathcal{L}, \mathcal{M}$ are invertible, then $\mathcal{L} \otimes \mathcal{M}$ is well (since the stalks are free). Thus, $\otimes$ gives us a binary operation on invertible sheaves. In fact, it gives it a group structure:

1. $\otimes$ is associative
2. $\mathcal{L} \otimes \mathcal{O} = \mathcal{L} = \mathcal{O} \otimes \mathcal{L}$
3. Set $\mathcal{L}^{-1} = \mathcal{L}^{\vee}$. Then $\mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{O}$ (By HW problem)

Def: The Picard group of X is the group of invertible sheaves (up to isomorphism) w/ operation $\otimes$.

Sheaves from divisors

Given a divisor on X, there is a natural associated sheaf. Hartshorne does this in full generality for Cartier divisors on arbitrary schemes, but we will focus on "nice" schemes where we have Weil divisors and loc. principal divisors.

For the remainder of the section, let X be a normal, separated, integral, Noetherian scheme w/ $K = K(X)$.

Def: Let D be a Weil divisor on X. The sheaf associated to D, denoted $\mathcal{O}_X(D)$ is defined

$$U \longmapsto \{ f \in K^* \mid \underbrace{(f) + D|_U \geq 0}_{\text{effective}} \} \cup \{0\}$$

This is a subsheaf of K - Check! Moreover, it's an $\mathcal{O}_X$-module:

If $f \in \mathcal{O}(U)$, $g \in \mathcal{O}(D)(U)$, then f is regular on U, so $(f) \geq 0$. Thus, $\underbrace{(fg)}_{= (f)+(g)} + D|_U \geq 0$, so $fg \in \mathcal{O}(D)(U)$.

If $D = \sum n_i Y_i$, then $\operatorname{Supp} D = \bigcup_{n_i \neq 0} Y_i$. Thus, on the open set $U = X \setminus \operatorname{Supp} D$,

$$\mathcal{O}(D)(U) = \{ f \mid (f) \geq 0 \}$$

i.e. f is regular on U. Thus, $\mathcal{O}(D)$ agrees w/ $\mathcal{O}$ away from the support of D.

We can give a different, but equivalent, definition of $\mathcal{O}(D)$ when D is Cartier (i.e. locally principal in this case):

Def: Let D be a Cartier divisor on X (normal, etc.), represented by $\{(U_i, f_i)\}$. Define $\mathcal{O}_X(D)$ as follows:

$$\mathcal{O}_X(D)\Big|_{U_i} = \frac{1}{f_i}\mathcal{O}_{U_i} \subseteq K$$

This is well-defined, since if $U \subseteq U_i \cap U_j$, $\frac{f_i}{f_j} \in \mathcal{O}_X^*(U)$, so

$$\frac{1}{f_i}\mathcal{O}_X(U) = \left(\frac{f_i}{f_j}\right)\left(\frac{1}{f_i}\mathcal{O}_X(U)\right) = \frac{1}{f_j}\mathcal{O}_X(U).$$

Thus, we have defined $\mathcal{O}(D)$ on a cover, so $\mathcal{O}(D)$ is the sheaf it generates.

Why are these definitions equivalent?

If $U \subseteq U_i \subseteq X$, w/ D defined by $\{(U_i, f_i)\}$ as above, then $D|_{U_i} = (f_i)$, so

$$\frac{1}{f_i}\mathcal{O}_X(U) \underset{\substack{\uparrow \\ \text{in } K}}{=} \left\{ f \in K^* \;\middle|\; (f) + D|_{U_i} \geq 0 \right\}$$ as follows:

Let $f \in K^*$. Then $(f) + D|_{U_i} \geq 0$

$$\iff (f) + (f_i) \geq 0$$

$$\iff (ff_i) \geq 0$$

$$\iff ff_i \text{ regular}$$

$$\iff f = \frac{1}{f_i} g \quad \text{with } g \in \mathcal{O}_X(U).$$

Thus, they are equal on a basis, so they must be the same sheaf.

Note that if $D = \{(U_i, f_i)\}$ Cartier, then

$$\mathcal{O}_{U_i} \cong \frac{1}{f_i}\mathcal{O}_{U_i} = \mathcal{O}_X(D)|_{U_i}$$
$$f \longleftrightarrow \frac{f}{f_i}$$

That is, $\mathcal{O}_X(D)$ is invertible.

In particular, if $D = (f)$ is principal, then $\mathcal{O}_X \cong \mathcal{O}_X(D)$. Thus, we have a well-defined function

$$\begin{aligned} \operatorname{CaCl} X &\longrightarrow \operatorname{Pic} X \\ D &\longmapsto \mathcal{O}_X(D). \end{aligned}$$

Prop: The map $\operatorname{CaCl} X \longrightarrow \operatorname{Pic} X$ is an isomorphism.

Pf: First we show it is a homomorphism:

Let $D_1 = \{(U_i, f_i)\}$, $D_2 = \{(U_i, g_i)\}$ be Cartier divisors (we can refine the corresponding covers so that they are the same).

Then $D_1 + D_2 = \{(U_i, f_i g_i)\}$. We have a map

$$\mathcal{O}(D_1) \otimes \mathcal{O}(D_2) \longrightarrow \mathcal{O}(D_1 + D_2)$$

defined on each U_i by $\frac{1}{f_i}\mathcal{O}_{U_i} \otimes \frac{1}{g_i}\mathcal{O}_{U_i} \to \frac{1}{f_i g_i}\mathcal{O}_{U_{ij}}$

which agrees on the overlaps, since

$$\begin{tikzcd}
\frac{1}{f_i}\mathcal{O}_{ij} \otimes \frac{1}{g_i}\mathcal{O}_{ij} \arrow[r] \arrow[d, "\cong"] & \frac{1}{f_i g_i}\mathcal{O}_{U_{ij}} \arrow[d, "\cong"] \\
\frac{1}{f_j}\mathcal{O}_{ij} \otimes \frac{1}{g_j}\mathcal{O}_{ij} \arrow[r] & \frac{1}{f_j g_j}\mathcal{O}_{U_{ij}}
\end{tikzcd}$$

commutes.

This is an isomorphism: on stalks, we have the inverse

$$\frac{1}{f_i g_i}\mathcal{O}_p \to \frac{1}{f_i}\mathcal{O}_p \otimes \frac{1}{g_i}\mathcal{O}_p$$
$$\frac{1}{f_i g_i} a \mapsto a\left(\frac{1}{f_i} \otimes \frac{1}{g_i}\right).$$

So $\mathcal{O}(D_1 + D_2) \cong \mathcal{O}(D_1) \otimes \mathcal{O}(D_2)$, and thus they are equal in $\operatorname{Pic} X$. Thus, the map is a homomorphism.

For surjectivity, we need that for $\mathcal{L}$ invertible, there is some divisor D s.t. $\mathcal{L} \cong \mathcal{O}_X(D)$.

First note that $\mathcal{L}$ injects into $\mathcal{K}$: We have a natural injection $\mathcal{L} \hookrightarrow \mathcal{L} \otimes \mathcal{K}$, and on any open set $U \subseteq X$ where $\mathcal{L} \otimes \mathcal{O}_X \cong \mathcal{O}_X$, we have $\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$. Thus, $\mathcal{L} \otimes \mathcal{K}$ is constant on a basis, so it must be a constant sheaf since X is irreducible. i.e. $\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$.

Thus we can assume $\mathcal{L}$ is a subsheaf of $\mathcal{K}$.

To recover D, cover X by $\{U_i\}$ s.t. $\mathcal{L}|_{U_i} \cong \mathcal{O}_{U_i}$.
Then $\mathcal{L}|_{U_i}$ is globally generated by some $f_i \in K^*$,

so $\mathcal{L}|_{U_i} = f_i \, \mathcal{O}_{U_i}$, so

$\mathcal{L} = \mathcal{O}_X(D)$, where $D = \{(U_i, 1/f_i)\}$, so the map is surjective.

In particular, if $\mathcal{O}(D) \cong \mathcal{O}_X$, we have $\mathcal{O}(D) = f\mathcal{O}_X$ for some $f \in K^*$, so $D = (1/f)$. Thus, the map is injective, so it's an isomorphism. $\square$

If X is additionally locally factorial, we showed $\mathrm{CaCl}\, X = \mathrm{Cl}\, X$, so we have:

Corollary: If X is Noetherian, integral, separated, locally factorial, then $\mathrm{Cl}\, X \cong \mathrm{Pic}\, X$. In particular, this holds for nonsingular varieties.

We can now explicitly describe all invertible sheaves on $\mathbb{P}^n_k$:

Cor: If $X = \mathbb{P}^n_k$, then every invertible sheaf is isomorphic to $\mathcal{O}_X(d)$ for some $d \in \mathbb{Z}$.

Pf: $\mathrm{Pic}\, X \cong \mathrm{Cl}\, X \cong \mathbb{Z}$, and the generator of $\mathrm{Cl}\, X$ is a hyperplane H. Thus, we want to show that $\mathcal{O}_X(H) \cong \mathcal{O}_X(1)$.

Take $S = k[x_0, \dots, x_n]$ and H cut out by $L = 0$, $L \neq x_i$. Set $U_i = D_+(x_i)$. Then

$H = \{(U_i, \frac{L}{x_i})\}$. i.e. $D|_{U_i} = (\frac{L}{x_i})$. Thus

$$\mathcal{O}(H)|_{U_i} = \frac{x_i}{L} \mathcal{O}_{U_i},$$ whose sections on U_i

are $\left\{ \frac{x_i}{L} \cdot \frac{F}{x_i^d} \,\middle|\, \deg F = d \right\} = \frac{1}{L} S(1)_{(x_i)}$

$$\frac{x_i}{L} \cdot \frac{F}{x_i^d} = \frac{1}{L} \cdot \frac{F}{x_i^{d-1}}$$

$$\Rightarrow \mathcal{O}_X(H) = \widetilde{\tfrac{1}{L} S(1)} \cong \widetilde{S(1)} = \mathcal{O}_X(1). \quad \square$$

Let D be an effective Cartier divisor on X (normal, etc.). Then $D = \{(U_i, f_i)\}$, $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$. The <u>associated subscheme</u> Y is the closed subscheme defined by the ideal sheaf locally generated by f_i.

Prop: With D and Y as above, $\mathcal{I}_Y \cong \mathcal{O}_X(-D)$.

Pf: $\mathcal{O}(-D) = f_i \mathcal{O}_{U_i}$ on each U_i, which is exactly the ideal sheaf $\mathcal{I}_Y$. $\square$

Cor: If $Y \subseteq \mathbb{P}^n_k$ is a degree d hypersurface, then $\mathcal{I}_Y = \mathcal{O}(-d)$.